Parallelization of Preconditioned MRTR Method Combined with Block-multicolor Ordering Supported by Level Structure Arising in RCM Ordering

Tomonori Tsuburaya[1, 2], Yoshifumi Okamoto[3], *Member*, *IEEE*, and Shuji Sato[4]

[1]Department of Innovation Systems Engineering, Utsunomiya University
7-1-2 Yoto, Utsunomiya, Tochigi 321-8585, Japan, dt147113@cc.utsunomiya-u.ac.jp
[2]Research Fellow of Japan Society for the Promotion of Science
[3]Department of Electronics and Electrical Engineering, Hosei University
3-7-2, Kajinocho, Koganei, Tokyo 184-8584, Japan, okamotoy@hosei.ac.jp
[4]Department of Electrical and Electronic Systems Engineering, Utsunomiya University
7-1-2 Yoto, Utsunomiya, Tochigi 321-8585, Japan, sato@cc.utsunomiya-u.ac.jp

The performance of preconditioned Minimized Residual method based on the Three-term Recurrence formula of the CG-type (MRTR) using multicolor (MC) ordering has been demonstrated on various symmetric linear systems derived from 3-D FEM. The elapsed time using MC is longer than that using block preconditioner with RCM ordering because the cache hit deteriorate owing to increase in bandwidth. To realize further reduction of elapsed time, the block-multicolor (BMC) ordering based on level structure arising in RCM ordering is proposed. The validity of proposed method is shown in comparison to conventional block preconditioner combined with the traditional orderings.

***Index Terms*—Finite element method, iterative methods, ordering technique, parallel processing.**

I. Introduction

THE parallelization of Krylov subspace method is essential for the fast electromagnetic field analysis. Recently, the performance of preconditioned MRTR method using multicolor (MC) ordering [1] has been demonstrated on real symmetric linear systems derived from 3-D FEM [2]. The forward and backward substitution can be completely parallelized by MC. However, the elapsed time using MC might be longer than the case using block preconditioner [3] with reverse Cuthill-McKee (RCM) ordering so that the cache hit deteriorate owing to increase in bandwidth.

Then, the block-multicolor (BMC) ordering [4] based on level structure arising in RCM ordering is proposed, and the abbreviation of the proposed method is RBMC. The number of synchronizations in forward and backward substitution is reduced so that all blocks can be colored with 2 colors. However, the load-balance in forward and backward substitution might deteriorate because the irregular block matrices are distributed around diagonal. To uniformize load-balance in forward and backward substitution, the procedure of RBMC is newly modified based on the concept of block red-black (BRB) ordering [5]. The modified ordering is abbreviated to modified RBMC. This paper shows the affinity of five orderings (RCM, MC, BMC, RBMC and modified RBMC) for parallelized MRTR method on real symmetric linear system arising in 3-D eddy current problem.

II. Proposed Ordering Technique

A. RBMC

Fig. 1 shows the example of RBMC ordering in graph representation. First, level structure is constructed by RCM as shown in Fig. 1 (b). Here, L1 ~ L9 denote the level number. The starting node is determined by node with minimum degree. Next, the level number is replaced with block number. The adjacent blocks are connected to each other. Consequently, all blocks can be colored with 2 colors. After coloring, the renumbering process is performed color by color. The unknowns in block are renumbered in order of increasing original unknown number shown in Fig. 1 (c).

The forward and backward substitution in RBMC can be parallelized color by color. It takes just one synchronization to achieve the parallelization of forward and backward substitution. However, RBMC has the disadvantage of poor load-balance because the number of unknowns in each level become non-uniform.

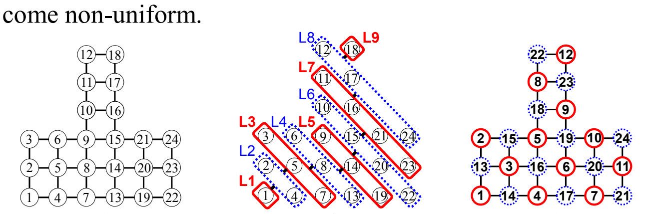

Fig. 1. RBMC ordering: (a) original, (b) blocking and coloring and (c) renumbering.

B. Modified RBMC

The load-balance in RBMC is improved by the concept of BRB ordering. In BRB ordering, the number of unknowns in block is previously defined. Then, blocking is performed so that all blocks can be colored with 2 colors. This strategy can be successfully introduced into modified RBMC. Fig. 2 shows the flowchart of modified RBMC. Here, n_b, s, $c(i)$, and $n(i)$ denote the number of blocks, number of unknowns per block, color number of block i, and the number of unknowns allocated to block i, respectively.

First, level structure is constructed by RCM. In the step 2, the n_b is determined by $n_b = 2N_p$, where N_p is the degree of parallelization. The s is calculated by s = DoF / n_b. If the DoF cannot be divided by n_b, remainder of nodes are added to final block n_b. Next, the allocation of block number is implemented by order of increasing level number in the step 4. Then, $c(i)$ is determined in the step 5. If the $c(i)$ is larger than 2, allocation of next block is performed. Otherwise, go on to step 7. If the $n(i)$ is equal to s, allocation of next block is performed. Otherwise, go on to next level. These processes are repeated until block number is allocated to all unknowns.

Fig. 3 shows the example of modified RBMC. Here, the N_p was set to 3. The new block number B1~B6 is allocated to all unknowns as shown in Fig. 3 (b). The unknowns in block are renumbered in order of increasing level number. When the level number is the same, the unknowns are renumbered in order of increasing original unknown number. Using new unknown number in Fig. 3 (c), the nonzero entries are rearranged as shown in Fig 4. In RBMC, the diagonal matrices are located in B_{11} and B_{22} because unknowns are defined on vertex of a quadrangle in graph representation. Generally, some block matrices are distributed in B_{11} and B_{22}. On the other hand, the diagonal block matrices R_i and B_i (i = 1, 2, 3) are arranged by modified RBMC. The number of block matrices in each color is equal to N_p. The forward and backward substitution can be parallelized by the same procedure as BMC.

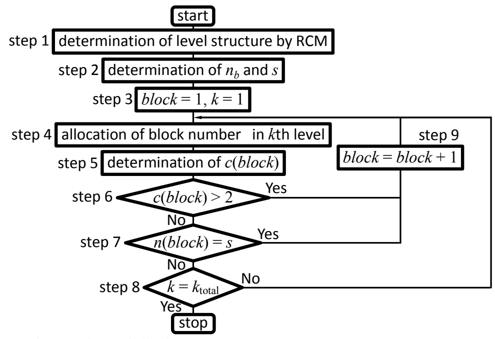

Fig. 2. Flowchart of modified RBMC.

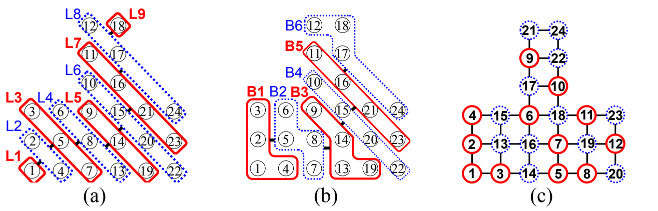

Fig. 3. Procedure of modified RBMC ordering: (a) level structure, (b) blocking and coloring and (c) renumbering.

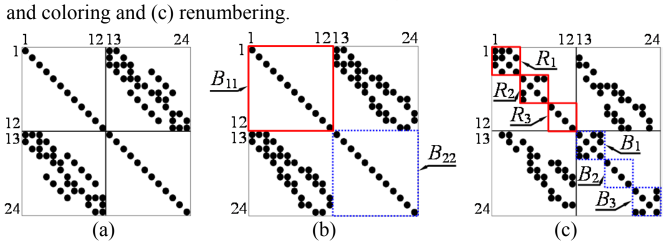

Fig. 4. Nonzero distributions: (a) original, (b) RBMC and (c) Modified RBMC.

III. Numerical Results

IEEJ eddy current model [2], which is discretized by 1st order edge-based tetrahedron (number of elements: 2,557,135, DoF: 2,983,657, number of nonzero entries: 48,711,313), is adopted as analysis model. The $\boldsymbol{A}$ formulation is applied to eddy current analysis. When the relative residual norm converged less than 10^{-6}, the iterative process is terminated.

Fig. 5 shows the nonzero distributions. In modified RBMC, N_p (= 6) block matrices are uniformly distributed in each color. When nonzero distributions of MC, BMC, RBMC and modified RBMC were utilized, the forward and backward substitution can be evaluated with all off-diagonal components. Table I lists the performance of parallelized MRTR with five orderings shown in Fig. 5. Here, s represents the number of nodes per block in BMC. T_{Np} denotes the total elapsed time using N_p threads. The convergence characteristic using BMC is superior to that using MC because the nonzero entries of BMC intend to be distributed around diagonal. The elapsed time using RBMC is shorter than that using MC and BMC according to reduction of bandwidth. Furthermore, the modified RBMC shows the highest efficacy for the reduction of elapsed time among all orderings because the convergence characteristic and load-balance in matrix-vector product, forward substitution and backward substitution could be improved. Hereafter, the performance of parallelized MRTR with Eisenstat's technique [2] by means of proposed method will be illustrated in the full paper.

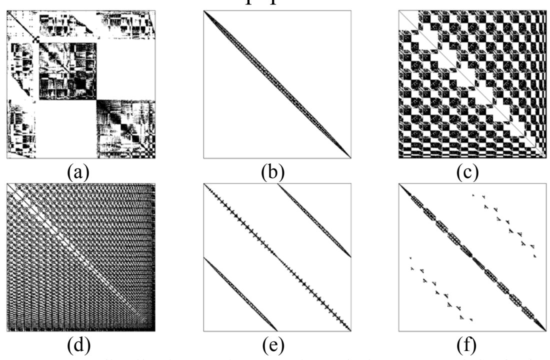

Fig. 5. Nonzero distributions using matrix orderings: (a) original, (b) RCM, (c) MC, (d) BMC(s), (e) RBMC and (f) Modified RBMC.

TABLE I Resultant Performance of Parallelized MRTR Method

precond.	ordering	N_p	linear it.	elapsed time [s] $\boldsymbol{Au}$	$L^{-1}\boldsymbol{u}$	$L^{-T}\boldsymbol{u}$	total (T_{Np})	scal. (T_1 / T_6)
Block SGS	RCM	1	230 (1.00)	12.1 (1.00)	10.2 (1.00)	8.7 (1.00)	39.4 (1.00)	2.3
		6	338 (1.46)	6.6 (0.54)	3.0 (0.29)	2.7 (0.31)	17.1 (0.43)	
SGS	MC 13 colors	1	266 (1.15)	45.1 (3.72)	24.0 (2.35)	28.7 (3.29)	108.0 (2.74)	3.6
		6	266 (1.15)	12.4 (1.02)	5.7 (0.55)	7.5 (0.86)	30.0 (0.76)	
	BMC(s) s = 512 55 colors	1	246 (1.06)	30.7 (2.53)	15.5 (1.51)	18.9 (2.17)	74.5 (1.89)	3.5
		6	246 (1.06)	8.0 (0.66)	4.0 (0.39)	4.9 (0.56)	21.0 (0.53)	
	RBMC 2 colors	1	253 (1.10)	14.5 (1.19)	10.5 (1.02)	12.6 (1.44)	47.2 (1.19)	3.0
		6	253 (1.10)	5.1 (0.42)	3.0 (0.29)	3.1 (0.35)	15.5 (0.39)	
	Modified RBMC 2 colors	1	230 (1.00)	12.0 (0.99)	9.8 (0.96)	11.5 (1.32)	42.0 (1.06)	3.1
		6	235 (1.02)	4.6 (0.38)	2.5 (0.24)	2.6 (0.29)	13.4 (0.33)	

$\boldsymbol{Au}$: matrix-vector product, $L^{-1}\boldsymbol{u}$: forward substitution, $L^{-T}\boldsymbol{u}$: backward substitution
CPU: Intel Xeon E5-2687W v2 (3.4 GHz), API for parallelization: OpenMP

IV. References

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